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Conjugation and Second-Order Properties of Convex Functions*

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We single out some second-order properties of convex functions that are well behaved with respect to the conjugacy operator. As an application, we prove that if a convex, lower semicontinuous function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ has a second order Taylor expansion at the origin

$$f(x) = \tfrac{1}{2}Ax \cdot x + o(|x|^2) \qquad \text{as} \quad x \to 0,$$

and the matrix A is symmetric and positive definite, then the conjugate function f^* has a second-order Taylor expansion at the origin, too, given by

$$f^*(y) = \tfrac{1}{2}A^{-1}y \cdot y + o(|y|^2) \qquad \text{as} \quad y \to 0.$$

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1. INTRODUCTION

The second-order differentiability of convex functions has been studied from many different points of view. The one we have adopted here is to find second-order properties that transform nicely when we turn from a convex function to its conjugate.

Let us start recalling some elementary notions from convex analysis (cf., e.g., [2, Chaps. 8 and 9]. Given a function $f\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, we define its *epigraph* epi f as

$$\operatorname{epi} f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : r \geqslant f(x)\}. \tag{1.1}$$

The function f is said to be proper if epi $f \neq \varnothing$, convex if epi f is a convex subset of $\mathbb{R}^n \times \mathbb{R}$, and it is lower semicontinuous if epi f is a closed set in $\mathbb{R}^n \times \mathbb{R}$.

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For any $y \in \mathbb{R}^n$ and a proper $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, consider the set I_y of the real numbers a such that the epigraph of the affine function $x \mapsto y \cdot x + a$ contains the epigraph of f (we indicate by $y \cdot x$ the canonical scalar product in $\mathbb{R}^n$). Of course, the set I_y is either empty or it is a closed interval with $-\infty$ as infimum. The function f^* which assigns to each y the opposite of the supremum of I_y ($\sup \emptyset = -\infty$) is the *convex conjugate* of f. Otherwise stated,

$$f^*(y) := \sup\{y \cdot x - f(x) : x \in \mathbb{R}^n\}. \tag{1.2}$$

The conjugate function f^* is convex and lower semicontinuous, because it is the supremum of a family of affine functions. Moreover, if f itself is proper, convex, and lower semicontinuous, then f^* is proper and its own conjugate f^{**} turns out to coincide with f (more generally, the epigraph of the biconjugate f^{**} is the convex closed hull of epi f). The conjugacy operator is an involution within the set of convex, lower semicontinuous functions on $\mathbb{R}^n$.

Another object attached to a convex, lower semicontinuous function f is the *subdifferential* which is the set ∂f of the couples $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$f(z) \geqslant f(x) + (z - x) \cdot y, \qquad \forall z \in \mathbb{R}^n. \tag{1.3}$$

Connection between f^* and ∂f are

$$(x, y) \in \partial f \Leftrightarrow (y, x) \in \partial f^* \Leftrightarrow f(x) + f^*(y) = x \cdot y. \tag{1.4}$$

If we regard ∂f as a relation in $\mathbb{R}^n$, then ∂f^* is simply the inverse relation. The symmetry of the subdifferential with respect to conjugation is perfect.

Conjugacy and subdifferentiation are defined in terms of the linear space structure only (for convex functions, even the lower semicontinuity can be rephrased as such, and the scalar product can be thought of as the coupling between $\mathbb{R}^n$ and its dual; but we will not insist on this point). However, there is a sizable intersection with the usual calculus notions for smooth functions on $\mathbb{R}^n$. For example, if $f: \mathbb{R}^n \to \mathbb{R}$ is C^1 and convex, then $(x, y) \in \partial f \Leftrightarrow y = \nabla f(x)$. If, moreover, $\nabla f: \mathbb{R}^n \to \mathbb{R}^n$ is invertible with continuous inverse, then f^* is C^1 and $\nabla f^* = (\nabla f)^{-1}$.

If f is convex, C^2, and the Hessian matrix $\nabla^2 f(x)$ is nonsingular in all points x, the inverse function theorem yields that $\nabla^2 f^*(\nabla f(x)) = (\nabla^2 f(x))^{-1}$. This last formula can be rewritten in terms of conjugation and subdifferential. Namely, let $D^2 f(x)$ be the quadratic form associated with the Hessian matrix, i.e., $(D^2 f(x))(z) := \frac{1}{2} \nabla^2 f(x)\, z \cdot z$. This quadratic form is a convex function of z. The formula becomes

$$(x, y) \in \partial f \Rightarrow D^2 f^*(y) = (D^2 f(x))^*, \tag{1.5}$$

that is to say, the second-order Taylor expansion of the conjugate of a function f is simply the conjugate of the second-order expansion of f (in the appropriate points).

The aim of the present work is to build upon f some kinds of second-order objects with the following properties:

(1) they are defined only in terms of the linear space structure;

(2) equalities akin to (1.5) come along naturally.

Suppose we are given a convex f and a couple $(\bar{x}, \bar{y}) \in \partial f$. The graph of the affine function $z \mapsto f(\bar{x}) + (z - \bar{x}) \cdot \bar{y}$ is tangent to the graph of f. By second-order information on f around the tangency point we usually mean the limit behaviour, as $t \searrow 0$, of the difference quotient

$$\Delta_{\bar{x}, \bar{y}, t} f(z) := \frac{f(\bar{x} + tz) - f(\bar{x}) - tz \cdot \bar{y}}{t^2}, \qquad z \in \mathbb{R}^n, \quad t > 0. \tag{1.6}$$

The idea followed by Rockafellar [6] is to study the limit, as $t \searrow 0$, of $\Delta_{\bar{x}, \bar{y}, t} f(\cdot)$ in the epi-convergence sense, in connection with the study of ∂f as a Lipschitzian manifold in $\mathbb{R}^n \times \mathbb{R}^n$. This is different from what we have in mind here, because it involves more topological structure than we are willing to do.

Hiriart-Urruty [3] builds his second-order subdifferential upon the *upper second-order directional derivative*

$$f''_+(\bar{x}, \bar{y}; z) := 2 \limsup_{t \searrow 0} \Delta_{\bar{x}, \bar{y}, t} f(z). \tag{1.7}$$

This function $f''_+(\bar{x}, \bar{y}; \cdot)$ vanishes in the origin, is nonnegative, convex, and positively homogeneous of degree 2. It may fail to be lower semicontinuous, though, as the following example (quoted from [3]) in $\mathbb{R}^2$ shows:

$$f(x_1, x_2) := \begin{cases} 0 & \text{in the circle } (x_1 + 1)^2 + x_2^2 \leqslant 1, \\ (x_1 + 1)^2 + x_2^2 - 1 & \text{elsewhere.} \end{cases} \tag{1.8}$$

We have $((0, 0), (0, 0)) \in \partial f$ and

$$f''_+((0, 0), (0, 0); (z_1, z_2)) = \begin{cases} 0 & \text{if } \ z_1 < 0, \\ 2z_2^2 & \text{if } \ z_1 = 0, \\ +\infty & \text{if } \ z_1 > 0. \end{cases} \tag{1.9}$$

However, by taking the convex closure we do not lose too much information. Good calculus rules can be proved (see Hiriart-Urruty and Seeger [4]).

The directional derivative f''_+ is also our starting point. Unfortunately, conjugation does not act symmetrically upon f''_+ itself, since it is not true in general that $(f^*)''_+ = (f''_+)^*$, even if we take the closure, as we will see in Example 4.3 of Section 4. In a way, what we try to do is to focus where the problem about conjugation lies. We will construct other entities that do transform neatly.

To simplify the formulas, up to a linear change of variables we can suppose $\bar{x} = \bar{y} = 0$ and $f(0) = 0$. Let $z \neq 0$ and set $\beta := f''_+(0, 0; z)$, assuming for now that $\beta < +\infty$. The relation

$$\beta = 2 \limsup_{t \searrow 0} \frac{f(tz)}{t^2} \tag{1.10}$$

only involves the value of f on the half-line $\{tz : t \geqslant 0\}$. Our first step is to rewrite (1.10) in terms of global (albeit trivial) inequalities between convex functions. Consider a function which is $+\infty$ on all of $\mathbb{R}^n$ except on a closed segment hinged at the origin and pointing in the direction z, where the function is quadratic:

$$\mathscr{E}^{z,a}(x) := \begin{cases} \frac{1}{2}t^2 & \text{if} \quad x = tz \text{ with } 0 \leqslant t \leqslant a, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.11}$$

The letter $\mathscr{E}$ is chosen as the initial of "elementary." The function $x \mapsto \mathscr{E}^{z,a}(x)$ is convex and lower semicontinuous. In terms of $\mathscr{E}$, formula (1.10) means that for every $\gamma > \beta$ there exists $a > 0$ such that

$$f \leqslant \gamma \mathscr{E}^{z,a} \qquad \text{on } \mathbb{R}^n. \tag{1.12}$$

Upon conjugation, the inequality reverses,

$$f^* \geqslant (\gamma \mathscr{E}^{z,a})^* \qquad \text{on } \mathbb{R}^n. \tag{1.13}$$

On the left-hand side, f^* is nonnegative, $f^*(0) = 0$, and $(0, 0) \in \partial f^*$. On the other side, there is a function $y \mapsto (\gamma \mathscr{E}^{z,a})^*(y)$ which is zero on the half-space $\{y \in \mathbb{R}^n : z \cdot y \leqslant 0\}$, it is $y \mapsto (z \cdot y)^2/2\gamma$ on the stripe $\{y \in \mathbb{R}^n : 0 \leqslant z \cdot y \leqslant a\gamma\}$, and it is affine on the half-space $\{y \in \mathbb{R}^n : z \cdot y \geqslant a\gamma\}$, the junctions being in C^1 fashion. When $n = 2$ we can figure out the graph of $(\gamma \mathscr{E}^{z,a})^*$ as a slice of cylindrical paraboloid, prolonged beyond the two edges with half-planes, in the smoothest possible way. In exact terms:

$$(\gamma \mathscr{E}^{z,a})^*(y) = \begin{cases} 0 & \text{if} \quad z \cdot y \leqslant 0, \\ (1/2\gamma)(z \cdot y)^2 & \text{if} \quad 0 \leqslant z \cdot y \leqslant a\gamma, \\ az \cdot y - \frac{1}{2}\gamma a^2 & \text{if} \quad z \cdot y \geqslant a\gamma. \end{cases} \tag{1.14}$$

While formula (1.12) is nontrivial only on the segment $\{tz : 0 \leqslant t \leqslant a\}$, it is equivalent to the relation (1.13) which gives nonobvious information on f^* on all of the half-space $\{y \in \mathbb{R}^n : z \cdot y > 0\}$. Vice versa, what must we know on f to obtain the value of $(f^*)''_+$? The answer is simply the set of all inequalities of the form $f \geqslant (\gamma \mathscr{E}^{z,a})^*$.

Here we are. The set of all the relations

$$\text{functions of (1.14) kind} \leqslant f \leqslant \text{functions of (1.11) kind} \tag{1.15}$$

is a package of second-order information on f at $(0, 0) \in \partial f$ that, under conjugation, transforms into the corresponding one for f^*, with no loss of knowledge. It allows us in particular to reconstruct f''_+ and $(f^*)''_+$, and it involves only the linear space structure.

The rest of the present article runs as follows. In Section 2 we make precise what we mean by formula (1.15). We hope we have provided enough motivation for it in this Introduction. We work out an $\mathscr{E}$-function machinery, that is meant to ease, as far as possible, second order calculations involving conjugates.

In Section 3 we give some comparison results with Hiriart-Urruty's and Rockafellar's approaches to the subject. In particular, Proposition 3.2 may suggest an alternative motivation for the $\mathscr{E}$-functions in terms of the approximate subdifferentials.

In Section 4 we develop a few examples in detail and prove some regularity results. We remark the following statement, which is a corollary of Proposition 4.5.

PROPOSITION 1.1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, convex function, with $f(0) = 0$, $f \geqslant 0$. Suppose that f has a second-order Taylor expansion at the origin,*

$$f(x) = \tfrac{1}{2} A x \cdot x + o(|x|^2) \qquad \text{as} \quad x \to 0, \tag{1.16}$$

where A is a symmetric, positive definite matrix. Then the conjugate function f^* has a second-order Taylor expansion at the origin, too, given by

$$f^*(y) = \tfrac{1}{2} A^{-1} y \cdot y + o(|y|^2) \qquad \text{as} \quad y \to 0. \tag{1.17}$$

Formula (1.17) was proved by Crouzeix [1] in the additional assumption that f be twice differentiable in all of a neighbourhood of 0, and $x \mapsto \nabla^2 f(x)$ be continuous at $x = 0$, with an argument involving the inverse function theorem (cf. also [5]). The proof we provide here uses directly formula (1.16) and computations of the conjugate of certain functions.

2. Basic Definitions and General Properties

Given vectors z, $y \in \mathbb{R}^n$ and a real number $a \geqslant 0$, we define the two following functions on $\mathbb{R}^n$:

$$\mathscr{E}^{z,a}(x) := \begin{cases} 0 & \text{if } \ x = 0, \\ \frac{1}{2}t^2 & \text{if } \ 0 \neq x = tz \text{ with } 0 < t \leqslant a, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

$$\mathscr{E}_{y,a}(x) := \begin{cases} 0 & \text{if } \ x \cdot y \leqslant 0, \\ \frac{1}{2}(x \cdot y)^2 & \text{if } \ 0 \leqslant x \cdot y \leqslant a, \\ ax \cdot y - \frac{1}{2}a^2 & \text{if } \ x \cdot y \geqslant a. \end{cases} \tag{2.2}$$

The first one is simply (1.11) and the second one is inspired by (1.14). The functions (2.1) and (2.3) will be called respectively upper and lower $\mathscr{E}$-functions.

The functions $\mathscr{E}^{z,a}$ and $\mathscr{E}_{z,a}$ are both proper, convex, and lower semicontinuous, the latter being also finite, C^1, and Lipschitz on $\mathbb{R}^n$ with constant $a\,|\,y\,|$. The correspondence between these functions and the parameters z, y, a is one-to-one, except when one of the parameters is zero. Actually, $\mathscr{E}^{z,0}$ and $\mathscr{E}^{0,a}$ are $+\infty$ everywhere except at the origin, where they vanish; $\mathscr{E}_{z,0}$ and $\mathscr{E}_{0,a}$ are zero throughout $\mathbb{R}^n$. If $z \neq 0$, $a \neq 0$, we can recover z from the value of $\mathscr{E}^{z,a}$ in a point $x \neq 0$, where the function is finite: $z = bx$, where $b^2\mathscr{E}^{z,a}(x) = \frac{1}{2}$. The value of a follows easily. The direction of $y \neq 0$ can be reconstructed by subdifferentiating $\mathscr{E}_{y,a}$ at any point, where it does not vanish. The actual value of y and a are then not difficult to find.

The following relations are straightforward: for $\gamma > 0$,

$$\begin{aligned} \gamma^2\mathscr{E}^{z,a} &= \mathscr{E}^{z/\gamma,\,\gamma a}, & \gamma^2\mathscr{E}_{y,a} &= \mathscr{E}_{\gamma y,\,\gamma a}, \\ \gamma^2\mathscr{E}^{z,a}\left(\frac{1}{\gamma}\,x\right) &= \mathscr{E}^{z,\,\gamma a}(x), & \gamma^2\mathscr{E}_{y,\,a}\left(\frac{1}{\gamma}\,x\right) &= \mathscr{E}_{y,\,\gamma a}(x); \end{aligned} \tag{2.3}$$

for $y \neq 0$, $z \neq 0$,

$$0 \leqslant a \leqslant b \Leftrightarrow \mathscr{E}^{z,a} \geqslant \mathscr{E}^{z,b}, \qquad \mathscr{E}_{y,a} \leqslant \mathscr{E}_{y,b} \tag{2.4}$$

(the implication $\Rightarrow$ is trivially true also for $y = 0$ or $z = 0$) and, unrestrictedly

$$\inf_{a\geqslant 0} \mathscr{E}^{z,a}(x)=\begin{cases}0 & \text{if } x=0,\\ \frac{1}{2}t^2 & \text{if } 0\neq x=tz \text{ with } t>0,\\ +\infty & \text{otherwise;}\end{cases} \tag{2.5}$$

$$\sup_{a\geqslant 0} \mathscr{E}_{y,a}(x)=\begin{cases}0 & \text{if } x\cdot y\leqslant 0,\\ \frac{1}{2}(x\cdot y)^2 & \text{if } x\cdot y\geqslant 0.\end{cases} \tag{2.6}$$

The $\mathscr{E}$-functions are designed to be easily conjugated, as we are going to see.

PROPOSITION 2.1. *The functions $\mathscr{E}^{z,a}$ and $\mathscr{E}_{z,a}$ are conjugates of each other.*

Proof. The subdifferential of $\mathscr{E}_{z,a}$ is readily computed, so that we can use for example characterization (1.4) and prove that

$$\mathscr{E}_{z,a}(x)+\mathscr{E}^{z,a}(y)=x\cdot y \Leftrightarrow y=\begin{cases}0 & \text{if } x\cdot z\leqslant 0,\\ (x\cdot z)\, z & \text{if } 0\leqslant x\cdot z\leqslant a,\\ az & \text{if } s\cdot z\geqslant a.\end{cases}$$

The equivalence holds trivially if $z=0$. Otherwise, the left-hand side is true if and only if $y=tz$ with $0\leqslant t\leqslant a$ and

$$\mathscr{E}_{z,a}(x)+\tfrac{1}{2}t^2=tx\cdot z,$$

that can be rewritten as

$$0=t^2-2(x\cdot z)\,t+\begin{cases}0 & \text{if } x\cdot z\leqslant 0,\\ (x\cdot z)^2 & \text{if } 0\leqslant x\cdot z\leqslant a,\\ 2a(x\cdot z)-a^2 & \text{if } s\cdot z\geqslant a.\end{cases}$$

This is an algebraic equation in t of degree 2, whose only solution in the interval $[0, a]$ is precisely

$$t=\begin{cases}0 & \text{if } x\cdot z\leqslant 0,\\ x\cdot z & \text{if } 0\leqslant x\cdot z\leqslant a,\\ a & \text{if } s\cdot z\geqslant a. \quad \blacksquare\end{cases}$$

It turns out that every convex, lower semicontinuous function, positively homogeneous of degree 2, is both the upper envelope of a family of lower $\mathscr{E}$-functions and the lower envelope of upper $\mathscr{E}$-functions (cf. also Proposition 4.4). For the latter claim, it suffices to note that such homogeneous function φ actually coincides with an upper $\mathscr{E}$-function on each segment $\{tx : 0 \leqslant t \leqslant a\}$. As for the former, just turn to the conjugate φ^*, that is homogeneous too, and remark that $\mathscr{E}_{y,a} \leqslant \varphi \Leftrightarrow (\mathscr{E}_{y,a})^* = \mathscr{E}^{y,a} \geqslant \varphi^*$; the upper envelope of $\{\mathscr{E}_{y,a} : \mathscr{E}_{y,a} \leqslant \varphi\}$ is lower semicontinuous, and if it did not coincide with φ, then φ^* would not coincide with the lower envelope of $\{\mathscr{E}^{y,a} : \mathscr{E}^{y,a} \geqslant \varphi^*\}$.

The neat conjugacy relations between $\mathscr{E}$-functions can then be exploited when dealing with conjugation of homogeneous functions, as we will do in the proof of Proposition 4.5.

PROPOSITION 2.2. *Given $x_0, y_0 \in \mathbb{R}^n$ with either $x_0 \cdot y_0 > 0$ or $y_0 = 0$, let S be the set of the convex, lower semicontinuous functions on $\mathbb{R}^n$, vanishing in 0, and whose subdifferential contains the segment in $\mathbb{R}^n \times \mathbb{R}^n$ joining $(0, 0)$ with (x_0, y_0). Then all functions of S coincide on $\{tx_0 : 0 \leqslant t \leqslant 1\}$, and S contains a smallest function, which is*

$$\frac{1}{x_0 \cdot y_0} \mathscr{E}_{y_0, x_0 \cdot y_0}$$

if $x_0 \cdot y_0 > 0$, and the zero function if $y_0 = 0$.

Proof. If $f \in S$, then the function $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by $g(t) := f(tx_0)$ is differentiable on $[0, 1]$ and $g'(t) = tx_0 \cdot y_0$. It follows that for all $t \in [0, 1]$

$$f(tx_0) = g(t) = \frac{x_0 \cdot y_0}{2} t^2 = \begin{cases} (1/x_0 \cdot y_0)\, \mathscr{E}_{y_0, x_0 \cdot y_0}(tx_0) & \text{if } \ x_0 \cdot y_0 > 0 \\ 0 & \text{if } \ y_0 = 0. \end{cases}$$

Again from $f \in S$, the epigraph of f is contained in the intersection, as $t \in [0, 1]$, of the half spaces

$$\{(x, r) \in \mathbb{R}^n \times \mathbb{R} : r \geqslant g(t) + (x - tx_0) \cdot ty_0\},$$

and this intersection is just the epigraph of $(1/x_0 \cdot y_0)\, \mathscr{E}_{y_0, x_0 \cdot y_0}$ if $x_0 \cdot y_0 > 0$, and of the zero function if $y_0 = 0$. ∎

The case $y_0 \neq 0$, $x_0 \cdot y_0 = 0$ would give rise to the non-C^1 function which is 0 on $\{z \in \mathbb{R}^n : z \cdot y_0 \leqslant 0\}$ and $z \mapsto z \cdot y_0$ elsewhere. The relation $x_0 \cdot y_0 < 0$ is not compatible with convexity. The reader can guess what characterization holds for $\mathscr{E}^{z, a}$.

The exact meaning we are going to attach to the inequality (1.15) is the following

DEFINITION 2.3. Given a proper, convex, and lower semicontinuous function $f: \mathbb{R}^n \to \mathbb{R}^n \cup \{+\infty\}$ and a point $(\bar{x}, \bar{y})$ of ∂f, we define the second-order lower and upper differentials of f at $(\bar{x}, \bar{y})$ as the following sets of functions:

$$\partial_-^2 f(\bar{x}, \bar{y}) := \{\mathscr{E}_{y,a} : y \in \mathbb{R}^n, a \geqslant 0, \forall x \in \mathbb{R}^n,$$
$$f(\bar{x}+x) \geqslant f(\bar{x}) + x \cdot \bar{y} + \mathscr{E}_{y,a}(x)\}, \qquad ((2.7)$$
$$\partial_+^2 f(\bar{x}, \bar{y}) := \{\mathscr{E}^{z,a} : z \in \mathbb{R}^n, a \geqslant 0, \forall x \in \mathbb{R}^n,$$
$$f(\bar{x}+x) \leqslant f(\bar{x}) + x \cdot \bar{y} + \mathscr{E}^{z,a}(x)\}. \qquad (2.8)$$

From the conjugacy relations among the $\mathscr{E}$-functions, we obtain at once the first of the promised links between the second-order properties of f and those of f^*.

PROPOSITION 2.4. *Let $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function, and let $(\bar{x}, \bar{y}) \in \partial f$. Then*

$$\partial_-^2 f^*(\bar{y}, \bar{x}) = (\partial_+^2 f(\bar{x}, \bar{y}))^*, \qquad \partial_+^2 f^*(\bar{y}, \bar{x}) = (\partial_-^2 f(\bar{x}, \bar{y}))^*. \quad (2.9)$$

(*If S is a set of functions, we indicate by S^* the set $\{g^* : g \in S\}$.*)

Proof. For the first equality, it suffices to note that the relation

$$f(\bar{x}+x) \leqslant f(\bar{x}) + x \cdot \bar{y} + \mathscr{E}^{z,a}(x), \qquad \forall x \in \mathbb{R}^n$$

is equivalent to the conjugate relation

$$f^*(\bar{y}+y) \geqslant f^*(\bar{y}) + \bar{x} \cdot y + \mathscr{E}_{z,a}(y), \qquad \forall y \in \mathbb{R}^n.$$

The second equality in (2.9) is just the first one as applied to f^* instead of f. ∎

If we defined $\partial^2 = \partial_-^2 \cup \partial_+^2$, then we could write $\partial^2 f^* = (\partial^2 f)^*$. But we prefer to preserve distinct notations.

The two objects ∂_-^2 and ∂_+^2 are quite complicated, as sets of functions. It may then be relieving to know that the information contained in them can be squeezed into two single functions, $\sup \partial_-^2 f(\bar{x}, \bar{y})$ and $\inf \partial_+^2 f(\bar{x}, \bar{y})$, where we mean pointwise extrema of families of functions. Supposing for simplicity that $\bar{x} = \bar{y} = 0$ and $f(0) = 0$, we can write

$$\sup \partial_-^2 f(0,0) \leqslant f \leqslant \inf \partial_+^2 f(0,0), \qquad (2.10)$$

whence we obtain

$$\begin{aligned} \mathscr{E}_{y,a} \in \partial_-^2 f(0,0) &\Leftrightarrow \mathscr{E}_{y,a} \leqslant \sup \partial_-^2 f(0,0), \\ \mathscr{E}^{z,a} \in \partial_+^2 f(0,0) &\Leftrightarrow \mathscr{E}^{z,a} \geqslant \inf \partial_+^2 f(0,0). \end{aligned} \qquad (2.11)$$

The function $\sup \partial^2_- f(\bar{x}, \bar{y})$ is convex and lower semicontinuous, because it is a supremum of a family of such functions. Somewhat surprisingly, also $\inf \partial^2_+ f(\bar{x}, \bar{y})$ is convex and lower semicontinuous. To prove the convexity, let $x_1, x_2 \in \mathbb{R}^n$, $\mathscr{E}^{z_1, a_1}$, $\mathscr{E}^{z_2, a_2} \in \partial^2_+ f(0, 0)$ be such that $\mathscr{E}^{z_1, a_1}(x_1)$ and $\mathscr{E}^{z_2, a_2}(x_2)$ are finite and close enough to the infima. Let t, $\theta \in [0, 1]$ (and again $f(0) = 0$),

$$\begin{aligned} f(t(\theta x_1 + (1-\theta) x_2)) &\leqslant \theta f(tx_1) + (1-\theta) f(tx_2) \\ &\leqslant \theta \mathscr{E}^{z_1, a_1}(tx_1) + (1-\theta) \mathscr{E}^{z_2, a_2}(tx_2) \\ &= t^2(\theta \mathscr{E}^{z_1, a_1}(x_1) + (1-\theta) \mathscr{E}^{z_2, a_2}(x_2)). \end{aligned}$$

From this we see that $2(\theta \mathscr{E}^{z_1, a_1}(x_1) + (1-\theta) \mathscr{E}^{z_2, a_2}(x_2)) \mathscr{E}^{\theta x_1 + (1-\theta) x_2, 1}$ is an element of $\partial^2_+ f(0, 0)$ (recall the first of formulas (2.3)), and its value in the point $\theta x_1 + (1-\theta) x_2$ is just what we needed. To prove the lower semicontinuity, let $x_n \to x \neq 0$ (the case $x = 0$ is obvious) and $\mathscr{E}^{z_n, a_n} \in \partial^2_+ f(0, 0)$ such that $\mathscr{E}^{z_n, a_n}(x_n) \to r \in [0, +\infty[$. Then, for all $t \in [0, 1]$,

$$f(tx_n) \leqslant \mathscr{E}^{z_n, a_n}(tx_n) = t^2 \mathscr{E}^{z_n, a_n}(x_n) \to t^2 r.$$

Since f is lower semicontinuous, we have $f(tx) \leqslant t^2 r$. This proves that r is greater than or equal to the value of $\inf \partial^2_+ f(0, 0)$ at x.

The relations (2.9) now imply

$$\begin{aligned} \sup \partial^2_- f^*(\bar{y}, \bar{x}) &= (\inf \partial^2_+ f(\bar{x}, \bar{y}))^*, \\ \inf \partial^2_+ f^*(\bar{y}, \bar{x}) &= (\sup \partial^2_- f(\bar{x}, \bar{y}))^*. \end{aligned} \tag{2.12}$$

Unfortunately these functions $\sup \partial^2_-$ and $\inf \partial^2_+$ lack the desirable property of being homogeneous of degree 2. That is why we are going to introduce two more functions, which are homogeneous without losing the conjugacy relation, at the cost of forsaking some of the information conveyed by ∂^2_- and ∂^2_+.

Definition 2.5. Let f and $(\bar{x}, \bar{y})$ be as in Definition 2.3. Let H be the set of the functions $\mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ that are nonnegative, proper, lower semicontinuous, and positively homogeneous of degree 2. Then we set, for all $w \in \mathbb{R}^n$,

$$\mathsf{D}^2_- f(\bar{x}, \bar{y}; w) := \inf\{\varphi(w) : \varphi \in H, \varphi \geqslant \mathscr{E}_{y,a}, \forall \mathscr{E}_{y,a} \in \partial^2_- f(\bar{x}, \bar{y})\}, \tag{2.13}$$

$$\mathsf{D}^2_+ f(\bar{x}, \bar{y}; w) := \sup\{\psi(\omega) : \psi \in H, \psi \leqslant \mathscr{E}^{z,a}, \forall \mathscr{E}^{z,a} \in \partial^2_+ f(\bar{x}, \bar{y})\}. \tag{2.14}$$

There are more explicit expressions for f''_+, $\mathsf{D}^2_- f$, and $\mathsf{D}^2_+ f$ in terms of $\partial^2_- f$ and $\partial^2_+ f$.

PROPOSITION 2.6. *Let f and $(\bar{x}, \bar{y})$ be as in Definition* 2.3. *Then* $\mathrm{D}^2_- f(\bar{x}, \bar{y}; \cdot)$ *and* $\mathrm{D}^2_+ f(\bar{x}, \bar{y}; \cdot)$ *are functions of* H. *Moreover,* $\mathrm{D}^2_- f(\bar{x}, \bar{y};\ ) \leqslant \mathrm{D}^2_+ f(\bar{x}, \bar{y}; \cdot) \leqslant \frac{1}{2} f''_+(\bar{x}, \bar{y}; \cdot)$ *and for all* $w \in \mathbb{R}^n$:

$$\tfrac{1}{2} f''_+(\bar{x}, \bar{y}; w) = \inf\{\mathscr{E}^{z, Ra}(w) : R > 0, \mathscr{E}^{z,a} \in \partial^2_+ f(\bar{x}, \bar{y})\}, \tag{2.15}$$

$$\mathrm{D}^2_+ f(\bar{x}, \bar{y}; \cdot) = (\textit{lower semicontinuous envelope of } \tfrac{1}{2} f''_+(\bar{x}, \bar{y}; \cdot)), \tag{2.16}$$

$$\mathrm{D}^2_- f(\bar{x}, \bar{y}; w) = \sup\{\mathscr{E}_{y, Ra}(w) : R > 0, \mathscr{E}_{y,a} \in \partial^2_- f(\bar{x}, \bar{y})\}. \tag{2.17}$$

Proof. $\mathrm{D}^2_+ f(\bar{x}, \bar{y}; \cdot)$ belongs to H because it is an upper envelope of a family of functions of H. Next, note that, if $\varphi \in H$, then

$$\begin{aligned}\mathscr{E}_{y,a} \leqslant \varphi \leqslant \mathscr{E}^{z,b} &\Leftrightarrow \forall R > 0,\ \mathscr{E}_{y,Ra} \leqslant \varphi \leqslant \mathscr{E}^{z,Rb} \\ &\Leftrightarrow \sup_{R>0} \mathscr{E}_{y,Ra} \leqslant \varphi \leqslant \inf_{R>0} \mathscr{E}^{z,Rb}.\end{aligned} \tag{2.18}$$

Observe next that the sup and inf in (2.18) are in H (see formulas (2.5) and (2.6)). The right-hand side of (2.17) is then in H, too. This settles (2.17) and the fact that $\mathrm{D}^2_- f(\bar{x}, \bar{y}; \cdot) \in H$. For (2.15) and the inequality $\mathrm{D}^2_- f(\bar{x}, \bar{y}; \cdot) \leqslant \frac{1}{2} f''_+(\bar{x}, \bar{y}; \cdot)$ just remind us of the definition of f''_+. To conclude with (2.16) recall that the lower semicontinuous envelope of a homogeneous function is still homogeneous. ∎

PROPOSITION 2.7. *Let f and $(\bar{x}, \bar{y})$ be as in Proposition* 2.3. *Then*

$$\mathrm{D}^2_- f^*(\bar{y}, \bar{x}; \cdot) = (\mathrm{D}^2_+ f(\bar{x}, \bar{y}; \cdot))^*, \qquad \mathrm{D}^2_+ f^*(\bar{y}, \bar{x}; \cdot) = (\mathrm{D}^2_- f(\bar{x}, \bar{y}; \cdot))^*. \tag{2.19}$$

Proof. We only need to note that, for a $\varphi \in H$, using (2.12),

$$\begin{array}{ccc}\sup \partial^2_- f(\bar{x}, \bar{y}) \leqslant \varphi \leqslant \inf \partial^2_+ f(\bar{x}, \bar{y}) & \Leftrightarrow & \mathrm{D}^2_- f(\bar{x}, \bar{y}; \cdot) \leqslant \varphi \leqslant \mathrm{D}^2_+ f(\bar{x}, \bar{y}; \cdot) \\ \Updownarrow & & \\ \sup \partial^2_- f^*(\bar{y}, \bar{x}) \leqslant \varphi^* \leqslant \inf \partial^2_+ f^*(\bar{y}, \bar{x}) & \Leftrightarrow & \mathrm{D}^2_- f^*(\bar{y}, \bar{x}; \cdot) \leqslant \varphi^* \leqslant \mathrm{D}^2_+ f^*(\bar{y}, \bar{x}; \cdot).\end{array}$$ ∎

Our paramount concern is with conjugation. However, we give the following calculus rule for the sum of two convex functions.

PROPOSITION 2.8. *Let $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semicontinuous functions, and suppose that $(\bar{x}, \bar{y}_f) \in \partial f$, $(\bar{x}, \bar{y}_g) \in \partial g$. Then*

$$\mathrm{D}^2_- f(\bar{x}, \bar{y}_f; \cdot) + \mathrm{D}^2_- g(\bar{x}, \bar{y}_g; \cdot) \leqslant \mathrm{D}^2_- (f+g)(\bar{x}, \bar{y}_f + \bar{y}_g; \cdot)\ . \tag{2.20}$$

Proof. It is easy to see that $(\bar{x}, \bar{y}_f + \bar{y}_g) \in \partial(f+g)$. Now pick

$$\mathscr{E}_{y_f, a_f} \in \partial^2_- f(\bar{x}, \bar{y}_f), \qquad \mathscr{E}_{y_g, a_g} \in \partial^2_- g(\bar{x}, \bar{y}_g). \tag{2.21}$$

The following inequality holds for all $x \in \mathbb{R}^n$:

$$(f+g)(\bar{x}+x) \geqslant (f+g)(\bar{x}) + x \cdot (\bar{y}_f + \bar{y}_g) + \underbrace{\mathscr{E}_{y_f, a_f}(x) + \mathscr{E}_{y_g, a_g}(x)}_{:=\,E(x)}. \tag{2.22}$$

The function E has the following property: for each $x_0 \in \mathbb{R}^n$ there exists $y_0 \in \mathbb{R}^n$, $a_0 > 0$ such that the subdifferential ∂E contains the segment $\{t(x_0, y_0) : 0 \leqslant t \leqslant a_0\}$, and either $x_0 \cdot y_0 > 0$ or $y_0 = 0$.

In the case $x_0 \cdot y_0 > 0$, from Proposition 2.2 and formulas (2.3) it follows that

$$E \geqslant \frac{1}{x_0 \cdot y_0} \mathscr{E}_{y_0, a_0 x_0 \cdot y_0},$$

and that the equality holds on $\{tx_0 : 0 \leqslant t \leqslant a_0\}$. Furthermore, we have from (2.22) that

$$\frac{1}{x_0 \cdot y_0} \mathscr{E}_{y_0, a_0 x_0 \cdot y_0} \in \partial^2_- (f+g)(\bar{x}, \bar{y}_f + \bar{y}_g).$$

In the case $y_0 = 0$ we have $E(tx_0) = 0$ for all $t \geqslant 0$. In either case, E coincides, on the segment $\{tx_0 : 0 \leqslant t \leqslant a_0\}$, with an element of $\partial^2_-(f+g)(\bar{x}, \bar{y}_f + \bar{y}_g)$.

To conclude we only need to remark that

$$\begin{aligned}
&\mathsf{D}^2_- f(\bar{x}, \bar{y}_f; x_0) + \mathsf{D}^2_- g(\bar{x}, \bar{y}_g; x_0) \\
&\quad = \sup\{\mathscr{E}_{y_f, R_1 a_f}(x_0) + \mathscr{E}_{y_g, R_2 a_g}(x_0) : R_1 > 0, R_2 > 0, \text{the rest as in (2.21)}\} \\
&\quad = \sup\{\mathscr{E}_{y_f, Ra_f}(x_0) + \mathscr{E}_{y_g, Ra_g}(x_0) : \ldots\} \\
&\quad = \sup\left\{\frac{1}{R^2} E(Rx_0) : R > 0, E \text{ as in (2.22)}\right\} \\
&\quad \leqslant \sup\left\{\frac{1}{R^2} \mathscr{E}_{y, a}(Rx_0) : R > 0, \mathscr{E}_{y, a} \in \partial^2_-(f+g)(\bar{x}, \bar{y}_f + \bar{y}_g)\right\} \\
&\quad = \mathsf{D}^2_-(f+g)(\bar{x}, \bar{y}_f + \bar{y}_g; x_0). \quad \blacksquare
\end{aligned}$$

The relation

$$(f+g)''_+(\bar{x}, \bar{y}_f + \bar{y}_g; \cdot) \leqslant f''_+(\bar{x}, \bar{y}_f; \cdot) + g''_+(\bar{x}, \bar{y}_g; \cdot) \tag{2.23}$$

is a consequence of the usual properties of limsup. It follows in particular that equality in both (2.20) and (2.23) occurs if we have $\mathrm{D}^2_- f = f''_+$ and $\mathrm{D}^2_- g = g''_+$. Unfortunately, these relations do not necessarily hold even if f and g are C^2 (see Example 4.7), unless restrictions on the Hessian matrices are imposed (Propositions 4.5 and 4.6).

3. Related Concepts

As we mentioned in the introduction, Hiriart-Urruty in [3] founds a second-order theory upon the the upper second-order directional derivative. He defines a second-order subdifferential of a convex, lower semicontinuous function $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at a point $(\bar{x}, \bar{y}) \in \partial f$ as

$$\partial^2 f(\bar{x}, \bar{y}) := \{ y \in \mathbb{R}^n : z \cdot y \leqslant (f''_+(\bar{x}, \bar{y}; z))^{1/2}, \forall z \in \mathbb{R}^n \}. \tag{3.1}$$

Within our framework, $\partial^2 f$ can be written also as follows.

Proposition 3.1. *Let $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous function, and let $(\bar{x}, \bar{y}) \in \partial f$. Then*

$$\begin{aligned} \partial^2 f(\bar{x}, \bar{y}) &= \{ y \in \mathbb{R}^n : 2\mathrm{D}^2_- f^*(\bar{y}, \bar{x}; y) \leqslant 1 \} \\ &= \textit{polar set of } \{ z \in \mathbb{R}^n : 2\mathrm{D}^2_+ f(\bar{x}, \bar{y}; z) \leqslant 1 \}. \end{aligned} \tag{3.2}$$

Proof. Using formulas (2.15) and (2.17),

$$\begin{aligned} y \in \partial^2 f(\bar{x}, \bar{y}) &\Leftrightarrow \forall z \in \mathbb{R}^n, z \cdot y \leqslant (2 f''_+(\bar{x}, \bar{y}; z))^{1/2} \\ &\Leftrightarrow \forall z \in \mathbb{R}^n, f''_+(\bar{x}, \bar{y}; z) \geqslant \begin{cases} \frac{1}{2}(z \cdot y)^2 & \text{if } z \cdot y \geqslant 0, \\ 0 & \text{if } z \cdot y < 0; \end{cases} \\ &\Leftrightarrow \forall a \geqslant 0, f''_+(\bar{x}, \bar{y}; \cdot) \geqslant \mathscr{E}_{y,a} \\ &\Leftrightarrow \forall a \geqslant 0, \mathrm{D}^2_+ f(\bar{x}, \bar{y}; \cdot) \geqslant \mathscr{E}_{y,a} \\ &\Leftrightarrow \forall a \geqslant 0, \mathrm{D}^2_- f^*(\bar{y}, \bar{x}; \cdot) \leqslant \mathscr{E}^{y,a} \\ &\Leftrightarrow 2\mathrm{D}^2_- f^*(\bar{y}, \bar{x}; y) \leqslant 1. \end{aligned}$$

The second expression in (3.2) follows if we note that $(2\mathrm{D}^2_+ f(\bar{x}, \bar{y}; \cdot))^{1/2}$ is the support function of the set $\{ z \in \mathbb{R}^n : 2 f''_+(\bar{x}, \bar{y}; z) \leqslant 1 \}$. ∎

Toward the end of [3], the author calls for a relation between $\partial^2 f^*(\bar{y}, \bar{x})$ and the polar set of $\partial^2 f(\bar{x}, \bar{y})$. Since $\partial^2 f^*(\bar{y}, \bar{x}) = \{ z \in \mathbb{R}^n :\ 2\mathrm{D}^2_- f(\bar{x}, \bar{y}; z)$

$\leqslant 1\}$, and the polar set of $\partial^2 f(\bar{x}, \bar{y})$ is $\{z \in \mathbb{R}^n : 2\mathsf{D}^2_+ f(\bar{x}, \bar{y}; z) \leqslant 1\}$, the inclusion

$$\partial^2 f^*(\bar{y}, \bar{x}) \supset \text{polar set of } \partial^2 f(\bar{x}, \bar{y}) \tag{3.3}$$

always holds, and a necessary and sufficient condition for the equality is

$$\mathsf{D}^2_- f(\bar{x}, \bar{y}; \cdot) \equiv \mathsf{D}^2_+ f(\bar{x}, \bar{y}; \cdot). \tag{3.4}$$

In the next section we will discuss some examples where (3.4) holds and others where it does not.

A second object with a very close connection with $\partial^2_\pm$ and $\mathsf{D}^2_\pm$ is the ε-subdifferential. To describe it, let us take a convex, lower semicontinuous function $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, a point $\bar{x} \in \mathbb{R}^n$, where $f(\bar{x}) < +\infty$, and $\varepsilon > 0$. The point $(\bar{x}, f(\bar{x}) - \varepsilon) \in \mathbb{R}^n \times \mathbb{R}$ does not belong to epi f. The convex closed hull of epi $f \cup \{(\bar{x}, f(\bar{x}) - \varepsilon)\}$ is the epigraph of a convex, lower semicontinuous function $\tilde{f}_\varepsilon$. The ε-subdifferential of f at $\bar{x}$ is just the (plain) subdifferential (multifunction) of $\tilde{f}_\varepsilon$ at $\bar{x}$,

$$\partial_\varepsilon f(\bar{x}) := \{y \in \mathbb{R}^n : f(\bar{x} + x) \geqslant f(\bar{x}) - \varepsilon + x \cdot y, \forall x \in \mathbb{R}^n\}. \tag{3.5}$$

The ε-subdifferential can also be expressed as a "slice set" of the conjugate function f^*,

$$\partial_\varepsilon f(\bar{x}) = \{y \in \mathbb{R}^n : f^*(y) \leqslant \bar{x} \cdot y + \varepsilon - f(\bar{x})\}. \tag{3.6}$$

As it was pointed out in [5], "$\partial_\varepsilon f(\bar{x})$ carries in a hidden form informations on the second-order behaviour of f around $\bar{x}$."

PROPOSITION 3.2. *Let $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous function, and let $(\bar{x}, \bar{y}) \in \partial f$, $y \in \mathbb{R}^n$, $a \geqslant 0$. Then*

$$\mathscr{E}_{y,a} \in \partial^2_- f(\bar{x}, \bar{y}) \Leftrightarrow \forall \varepsilon \in [0, a^2/2], \qquad \bar{y} + y\sqrt{2\varepsilon} \in \partial_\varepsilon f(\bar{x}). \tag{3.7}$$

Proof. Suppose $\mathscr{E}_{y,a} \in \partial^2_- f(\bar{x}, \bar{y})$. Then, for every $x \in \mathbb{R}^n$, $0 \leqslant x \cdot y \leqslant a$,

$$f(\bar{x} + x) \geqslant f(\bar{x}) + x \cdot \bar{y} + \mathscr{E}_{y,a}(x) := g(x).$$

The function g is continuous and convex. Take $\varepsilon \in [0, a^2/2]$ and $x_\varepsilon \in \mathbb{R}^n$ such that $x_\varepsilon \cdot y = \sqrt{2\varepsilon}$. Subdifferentiating g in x_ε we obtain

$$\begin{aligned} f(\bar{x} + x) &\geqslant g(x) \geqslant g(x_\varepsilon) + (x_\varepsilon \cdot y)((x - x_\varepsilon) \cdot y) \\ &= f(\bar{x}) - \varepsilon + x \cdot (\bar{y} + y\sqrt{2\varepsilon}), \end{aligned}$$

i.e., $\bar{y} + y\sqrt{2\varepsilon} \in \partial_\varepsilon f(\bar{x})$. Vice versa, suppose that $\bar{y} + y\sqrt{2\varepsilon} \in \partial_\varepsilon f(\bar{x})$ for all

$\varepsilon \in [0, a^2/2]$. Then epi f is contained in the intersection, over $\varepsilon \in [0, a^2/2]$, of the half-spaces

$$\{(x, r) \in \mathbb{R}^n \times \mathbb{R} : r \geqslant f(\bar{x}) - \varepsilon + (x - \bar{x}) \cdot (\bar{y} + y\sqrt{2\varepsilon})\},$$

and this intersection is simply the epigraph of $x \mapsto f(\bar{x}) + (x - \bar{x}) \cdot \bar{y} + \mathscr{E}_{y,a}(x - \bar{x})$. The proof could also be carried out reasoning on the conjugate f^* and the functions $\mathscr{E}^{y,a}$, using the characterization (3.6). ∎

Some approaches to second-order generalized differentiability of convex functions are based on set-convergence of certain sets associated with the function. Suppose that $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex, lower semicontinuous function, and, to simplify formulas, that $f(0) = 0$, $f \geqslant 0$. Recall the difference quotient (1.6) in the preset context

$$\Delta_{0,0,t} f(w) := \frac{f(tw)}{t^2}. \tag{3.8}$$

Rockafellar in [6] considers the epi-convergence of $\Delta_{0,0,t} f$ as $t \searrow 0$, that is, the set-convergence of epi $\Delta_{0,0,t} f$. Hiriart-Urruty and Seeger in [5] study the Dupin indicatrices of f at $(0, 0) \in \partial f$, which are the lim inf, lim sup, and limit, in the set-convergence sense, of the slice sets

$$\Sigma_t := \{w \in \mathbb{R}^n : \Delta_{0,0,t} f(w) \leqslant \tfrac{1}{2}\}. \tag{3.9}$$

For all $\mathscr{E}_{y,a} \in \partial^2_- f(0, 0)$, $\mathscr{E}^{z,b} \in \partial^2_+ f(0, 0)$, $t > 0$, and $w \in \mathbb{R}^n$ we have

$$\mathscr{E}_{y,a}(tw) \leqslant t^2 \Delta_{0,0,t} f(w) \leqslant \mathscr{E}^{z,b}(tw). \tag{3.10}$$

Using formulas (2.3), we obtain $\mathscr{E}_{y,a/t} \leqslant \Delta_{0,0,t} \leqslant \Delta_{0,0,t} f \leqslant \mathscr{E}^{z,b/t}$. In this inequality, the first function increases and the last one decreases when t decreases, so that

$$0 < t \leqslant 1/R \Rightarrow \mathscr{E}_{y,Ra} \leqslant \Delta_{0,0,t} f \leqslant \mathscr{E}^{z,Rb}, \tag{3.11}$$

that is to say,

$$0 < t \leqslant 1/R \Rightarrow \text{epi } \mathscr{E}^{z,Rb} \subset \text{epi } \Delta_{0,0,t} f \subset \text{epi } \mathscr{E}_{y,Ra}. \tag{3.12}$$

As a consequence, ("cl" meaning closure)

$$\begin{aligned} \text{epi } \mathrm{D}^2_+ f(0, 0; \cdot) = \text{cl epi} f''_+(0, 0; \cdot) &= \text{cl} \bigcup_{\mathscr{E}^{z,b} \in \partial^2 f(0,0),\ R > 0} \text{epi } \mathscr{E}^{z,Rb} \\ &\subset \liminf_{t \searrow 0} \text{epi } \Delta_{0,0,t} f \subset \limsup_{t \searrow 0} \text{epi } \Delta_{0,0,t} f \\ &\subset \bigcap_{\mathscr{E}_{y,a} \in \partial^2_- f(0,0), R > 0} \text{epi } \mathscr{E}_{y,Ra} = \text{epi } \mathrm{D}^2_- f(0, 0; \cdot), \end{aligned}$$

and

$$\{w \in \mathbb{R}^n : \mathsf{D}^2_+ f(0,0;w) \leqslant \tfrac{1}{2}\} \subset \liminf_{t \searrow 0} \Sigma_t$$
$$\subset \limsup_{t \searrow 0} \Sigma_t \subset \{w \in \mathbb{R}^n : \mathsf{D}^2_- f(0,0;w) \leqslant \tfrac{1}{2}\}.$$

A sufficient condition for the existence of both the epi-limit of $\Delta_{0,0,t} f$ and of the limit-Dupin indicatrix is again the equality (3.4), for $(\bar{x}, \bar{y}) = (0, 0)$.

The next formula follows the same way:

$$\mathsf{D}^2_- f(0,0;w) \leqslant \left\{ \begin{matrix} \liminf_{t \searrow 0,\, w' \to w} \Delta_{0,0,t} f(w') \\ \mathsf{D}^2_+ f(0,0;w) \end{matrix} \right\} \leqslant \limsup_{t \searrow 0} \Delta_{0,0,t} f(w) = \tfrac{1}{2} f''_+(\bar{x}, \bar{y}; w). \tag{3.13}$$

Each of these inequalities can be strict, as we see in the next section.

4. Examples

EXAMPLE 4.1. The absolute value $x \mapsto |x|$ in $\mathbb{R}$ shows that the functions $\mathsf{D}^2_- f$ and $\mathsf{D}^2_+ f$ need not be finite. Our second-order objects can all be explicitely written in $(0, 0) \in \partial\, |\cdot|$:

$$\partial^2_- \, |\cdot|(0,0) = \{\mathscr{E}_{y,a} : y \neq 0,\ 0 \leqslant a \leqslant 1/|y|\},$$
$$\partial^2_+ \, |\cdot|(0,0) = \{\mathscr{E}^{0,0}\},$$
$$\mathsf{D}^2_- \, |\cdot|(0,0;w) = \mathsf{D}^2_+ \, |\cdot|(0,0;w) = |\cdot|''_+ (0,0;w) = \begin{cases} 0 & \text{if } \ w = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

In this case it is especially easy to check the conjugacy relations (2.9) and (2.18).

EXAMPLE 4.2. For the function (1.8) mentioned in the introduction, we have

$$\mathsf{D}^2_- f(\bar{x}, \bar{y}; \cdot) = \mathsf{D}^2_+ f(\bar{x}, \bar{y}; \cdot) \not\equiv \liminf_{t \searrow 0} \Delta_{\bar{x}, \bar{y}, t} = \limsup_{t \searrow 0} \Delta_{\bar{x}, \bar{y}, t} = f''_+(\bar{x}, \bar{y}; \cdot),$$

when $\bar{x} = \bar{y} = (0, 0) \in \mathbb{R}^2$. In more details,

$$\partial^2_- f(\bar{x}, \bar{y}) = \{\mathscr{E}_{y,a} : y = (y_1, 0) \in \mathbb{R}^2,\ 0 \leqslant y_1 \leqslant \sqrt{2},\ a \geqslant 0\}$$
$$\cup \{\mathscr{E}_{y,a} : y = (y_1, 0) \in \mathbb{R}^2,\ y_1 > \sqrt{2},$$
$$0 \leqslant a \leqslant y_1(1 + (1 - ((1/y_1^2) - (1/2))^2)^{1/2})\},$$

$$\partial^2_+ f(\bar{x}, \bar{y}) \supset \{\mathscr{E}^{z,a} : z = (0, z_2) \in \mathbb{R}^2, |z_2| \leqslant 1/\sqrt{2}, a \geqslant 0\}$$
$$\cup \{\mathscr{E}^{z,a} : z = (z_1, z_2) \in \mathbb{R}^2, z_1 < 0, (az_1 + 1)^2 + a^2 z_2^2 \leqslant 1\},$$

$$\mathsf{D}^2_- f(\bar{x}, \bar{y}; w) = \mathsf{D}^2_+ f(\bar{x}, \bar{y}; w) = \begin{cases} 0 & \text{if} \quad w = (w_1, w_2) \in \mathbb{R}^2, w_1 \leqslant 0 \\ +\infty & \text{otherwise.} \end{cases}$$

EXAMPLE 4.3. We are going to describe a convex, continuous function $f: \mathbb{R} \to \mathbb{R}$ such that $(0, 0) \in \partial f$ and

$$\mathsf{D}^2_- f(0, 0; \cdot) = \liminf_{t \searrow 0} \Delta_{0,0,t} \not\equiv \mathsf{D}^2_+ f(0, 0; \cdot) = f''_+(0, 0; \cdot).$$

Consider the two parabolas $y_1 = x^2$ and $y_2 = 2x^2$. Take the point $(1, 1)$ on the former and draw the two straight lines tangent to the latter. Each of them meets the parabola $y_1 = x^2$ in one further point. Repeat the procedure from these points, drawing the new straight lines tangent to $y_2 = 2x^2$, and so forth. On one side the intersection points approach the origin, on the other they flee away to infinity. Add the symmetric of everything with respect to the axis $x = 0$. All the resulting straight lines bundle up the epigraph of a continuous convex function f. In every neighbourhood of the origin the graph of f intersects each of the two parabolas in infinitely many points. A computation leads to

$$\partial^2_- f(0, 0) = \{\mathscr{E}_{y,a} : |y| \leqslant \sqrt{2}, a \geqslant 0\},$$
$$\partial^2_+ f(0, 0) = \{\mathscr{E}^{z,a} : |z| \leqslant \tfrac{1}{2}, a \geqslant 0\},$$
$$\mathsf{D}^2_- f(0, 0; w) = \liminf_{t \searrow 0} \Delta_{0,0,t}(w) = w^2,$$
$$\mathsf{D}^2_+ f(0, 0; w) = f''_+(0, 0; w) = 2w^2.$$

After three examples of nonsmoothness, what can we say about $\partial^2_\pm$, $\mathsf{D}^2_\pm$ of functions that are smooth in senses not tailored for convexity? The first of the instances we are able to handle is the homogeneous functions.

PROPOSITION 4.4. *Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, convex function, positively homogeneous of degree 2, with $\varphi(0) = 0$, $\varphi \geqslant 0$. Then φ^* is positively homogeneous of degree 2, and*

$$\partial^2_+ \varphi(0, 0) = \{\mathscr{E}^{z,a} : \varphi(z) \leqslant \tfrac{1}{2}, a \geqslant 0\}, \tag{4.1}$$
$$\partial^2_- \varphi(0, 0) = \{\mathscr{E}_{y,a} : \varphi^*(y) \leqslant \tfrac{1}{2}, a \geqslant 0\}, \tag{4.2}$$
$$\mathsf{D}^2_- \varphi(0, 0; \cdot) = \varphi = \mathsf{D}^2_+ \varphi(0, 0; \cdot) = \varphi''_+(0, 0; \cdot), \tag{4.3}$$
$$\mathsf{D}^2_+ \varphi^*(0, 0; \cdot) = \varphi^* = \mathsf{D}^2_- \varphi^*(0, 0; \cdot) = (\varphi^*)''_+(0, 0; \cdot). \tag{4.4}$$

Proof. From the definition of φ^* we obtain at once that it is homogeneous. As for (4.1), if $a>0$

$$\begin{aligned} \mathscr{E}^{z,a} \in \partial_+^2 \varphi(0,0) \Leftrightarrow \mathscr{E}^{z,a} \geqslant \varphi \Leftrightarrow \varphi(tz) \leqslant t^2/2, \qquad \forall t \in [0,a] \\ \Leftrightarrow \varphi(z) \leqslant \tfrac{1}{2}. \end{aligned} \tag{4.5}$$

Formula (4.2) is just (4.1) applied to φ^*, using the equivalence $\mathscr{E}_{y,a} \in \partial_-^2 \varphi(0,0) \Leftrightarrow \mathscr{E}^{y,a} \in \partial_+^2 \varphi^*(0,0)$. The next two formulas are easy consequences. ∎

The second regular case is when f has positive definite second-order Fréchet derivative.

PROPOSITION 4.5. *Let f, $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous, convex functions, with φ positively homogeneous of degree 2, and such that*

$$\frac{c}{2}|x|^2 \leqslant \varphi(x) \leqslant \frac{c'}{2}|x|^2, \qquad \forall x \in \mathbb{R}^n \tag{4.6}$$

for two constants $c' \geqslant c > 0$. Then the following two conditions are equivalent

$$f(x) = \varphi(x) + o(|x|^2) \qquad \textit{as} \quad x \to 0, \tag{4.7}$$

$$f^*(y) = \varphi^*(y) + o(|y|^2) \qquad \textit{as} \quad y \to 0, \tag{4.8}$$

and they imply the formulas

$$\begin{aligned} \mathsf{D}_-^2 f(0,0;x) = \varphi(x) = \mathsf{D}_+^2 f(0,0;x) = f''_+(0,0;x), \\ \mathsf{D}_-^2 f^*(0,0;y) = \varphi^*(y) = \mathsf{D}_+^2 f^*(0,0;y) = (f^*)''_+(0,0;y). \end{aligned} \tag{4.9}$$

Proof. As a consequence of (4.6) we can write

$$\frac{1}{2c'}|y|^2 \leqslant \varphi^*(y) \leqslant \frac{1}{2c}|y|^2 \tag{4.10}$$

so that the statement of the proposition is perfectly symmetrical with respect to conjugation. Remark that if formulas (4.7) and (4.8) hold, then in particular f and f^* are first-order Fréchet differentiable at the origin, with zero differential. Then assume formula (4.7) and let us set out to prove (4.8). Let $0 < \varepsilon < 1$ be fixed, and define

$$\varphi_1(x) := (1-\varepsilon)\varphi(x), \qquad \varphi_2(x) := (1+\varepsilon)\varphi(x). \tag{4.11}$$

By conjugation

$$\varphi_1^*(y) = \frac{1}{1-\varepsilon}\,\varphi^*(y), \qquad \varphi_2^*(y) = \frac{1}{1+\varepsilon}\,\varphi^*(y). \tag{4.12}$$

From (4.6) and (4.7) we obtain a $\delta > 0$ such that

$$|x| \leqslant \delta \Rightarrow \varphi_1(x) \leqslant f(x) \leqslant \varphi_2(x). \tag{4.13}$$

We have $f(0) = 0$ and $x \neq 0 \Rightarrow f(x) > 0$, so that $\partial f^*(0) = \{0\}$, in the multifunction sense. Since the space is finite-dimensional, the subdifferential multifunctions are norm continuous at the points, where they are single-valued. Hence there exists $\gamma_1 > 0$ such that, using also (1.2) and (1.4),

$$\begin{aligned} |y| \leqslant \gamma_1 &\Rightarrow |x| \leqslant \delta, \forall x \in \partial f^*(y) \\ &\Rightarrow \varphi_1(x) \leqslant f(x) = x \cdot y - f^*(y), \forall x \in \partial f^*(y) \\ &\Rightarrow f^*(y) \leqslant x \cdot y - \varphi_1(x) \leqslant \varphi_1^*(y), \forall x \in \partial f^*(y) \\ &\Rightarrow f^*(y) \leqslant \varphi_1^*(y). \end{aligned}$$

Analogously, since $\partial\varphi_2^*(0) = \{0\}$, there exists $\gamma_2 > 0$ such that

$$|y| \leqslant \gamma_2 \Rightarrow \varphi_2^*(y) \leqslant f^*(y).$$

Summing up, there is $\gamma > 0$ such that

$$|y| \leqslant \gamma \Rightarrow \varphi_2^*(y) \leqslant f^*(y) \leqslant \varphi_1^*(y),$$

and this settles formula (4.8) because of (4.10). Formulas (4.9) follow easily.

Actually, the fact that the space is finite-dimensional is inessential. What we are going to show next is that

$$|y| \leqslant (1-\varepsilon)\, c\delta/2 \Rightarrow \varphi_2^*(y) \leqslant f^*(y) \leqslant \varphi_1^*(y), \tag{4.14}$$

via normed space structure and $\mathscr{E}$-function machinery.

(i) Using Proposition 4.4 we can write

$$\begin{aligned} \varphi_2(z) \leqslant \tfrac{1}{2} &\Leftrightarrow \forall a \geqslant 0, \mathscr{E}^{z,a} \in \partial_+^2\, \varphi_2(0,0) \\ &\Leftrightarrow \forall a \geqslant 0, \mathscr{E}_{z,a} \in \partial_-^2\, \varphi_2^*(0,0). \end{aligned}$$

Let $z \neq 0$. Then

$$\begin{aligned} \varphi_2(z) \leqslant \tfrac{1}{2} &\Rightarrow \forall a \geqslant 0, \mathscr{E}^{z,a} \geqslant \varphi_2 \\ &\Rightarrow \mathscr{E}^{z,\delta/|z|} \geqslant f \Leftrightarrow \mathscr{E}_{z,\delta/|z|} \leqslant f^*. \end{aligned}$$

Since

$$\varphi_2(z) \leqslant \frac{1}{2} \Rightarrow |z|^2 \leqslant \frac{1}{(1+\varepsilon)\,c},$$

we have

$$\begin{aligned} |y| \leqslant (1+\varepsilon)\,c\delta \Rightarrow y \cdot z \quad &\leqslant \delta/|z|,\ \forall z \neq 0 \text{ such that } \varphi_2(z) \leqslant \tfrac{1}{2} \\ &\Rightarrow f^*(y) \geqslant \sup\{\mathscr{E}_{z,\delta/|z|}(y) : z \neq 0,\ \varphi_2(z) \leqslant \tfrac{1}{2}\} \\ &= \sup\{\mathscr{E}_{z,a}(y) : z \neq 0,\ a \geqslant 0,\ \varphi_2(z) \leqslant \tfrac{1}{2}\} \\ &= \mathrm{D}^2_-\ \varphi_2^*(0, 0; y) = \varphi_2^*(y), \end{aligned}$$

and half of (4.14) is proved.

(ii) Since f is convex, $f(0) = 0$, and from (4.13), we have

$$|x| \geqslant \delta \Rightarrow f(x) \geqslant (1-\varepsilon)\,c\delta\,|x|/2. \tag{4.15}$$

This implies that if $\mathscr{E}_{y,a}(x) \leqslant \varphi_2(x)$ on the set $\{x \in \mathbb{R}^n : |x| \leqslant \delta\}$ and if the norm of the gradient of $\mathscr{E}_{y,a}$ does not exceed $(1-\varepsilon)\,c\delta/2$, then $\mathscr{E}_{y,a} \leqslant f$ on all of $\mathbb{R}^n$. If $y \neq 0$, again from Proposition 4.4 we can write

$$\begin{aligned} \varphi_1^*(y) \leqslant \tfrac{1}{2} &\Leftrightarrow \forall a \geqslant 0,\ \mathscr{E}^{y,a} \in \partial^2_+\ \varphi_1^*(0, 0) \\ &\Leftrightarrow \forall a \geqslant 0,\ \mathscr{E}_{y,a} \in \partial^2_-\ \varphi_1(0, 0) \end{aligned}$$

so that

$$\begin{aligned} \varphi_1^*(y) \leqslant \tfrac{1}{2} &\Rightarrow \forall a \geqslant 0,\ \mathscr{E}_{y,a} \leqslant \varphi_1 \\ &\Rightarrow \mathscr{E}_{y,(1-\varepsilon)c\delta/2|y|} \leqslant f \Leftrightarrow \mathscr{E}^{y,(1-\varepsilon)c\delta/2|y|} \geqslant f^*. \end{aligned}$$

On the other hand, if $y \neq 0$,

$$|y| \leqslant (1-\varepsilon)\,c\delta/2 \Rightarrow y = t\frac{y}{t} \qquad \text{with} \quad 0 < t \leqslant \frac{(1-\varepsilon)\,c\delta}{2\,|y/t|},$$

so that we have, again for $y \neq 0$,

$$\begin{aligned} &|y| \leqslant (1-\varepsilon)\,c\delta/2 \Rightarrow f^*(y) \\ &\quad \leqslant \inf\{\mathscr{E}^{y',(1-\varepsilon)c\delta/2|y'|}(y) : y' \neq 0,\ \varphi_1^*(y') \leqslant \tfrac{1}{2}\} \\ &\quad = \inf\{\mathscr{E}^{y',(1-\varepsilon)c\delta/2|y'|}(y) : y' = y/t,\ t > 0,\ \varphi_1^*(y') \leqslant \tfrac{1}{2}\} \\ &\quad = \inf\{\mathscr{E}^{y',a}(y) : y' = y/t,\ t > 0,\ a \geqslant 0,\ \varphi_1^*(y') \leqslant \tfrac{1}{2}\} \\ &\quad = \mathrm{D}^2_+\ \varphi_1^*(0, 0; y) = \varphi_1^*(y), \end{aligned}$$

and the proof of (4.14) is complete. ∎

The f in the proposition does not need to be differentiable in a neighbourhood of the origin, and φ is not necessarily of the form $\varphi(x)=\frac{1}{2}Ax\cdot x$, with A a symmetric, positive definite matrix. To see this, we can repeat the construction of Example 4.3 with the curves

$$y_1=\begin{cases} x^2 & \text{if} \quad x\geqslant 0, \\ 2x^2 & \text{if} \quad x\leqslant 0, \end{cases}$$

and $y_2=y_1+x^4$, instead of the two parabolas.

We are able to weaken the assumptions on φ if we know enough about f on its slice sets. The proof is very similar to the previous one.

PROPOSITION 4.6. *Let f, $\varphi:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ be two convex, lower semi-continuous functions, with $f(0)=\varphi(0)=0$, $f\geqslant 0$, $\varphi\geqslant 0$, and φ continuous and positively homogeneous of degree 2. Suppose that*

$$f''_+(0,0;\cdot)\equiv\varphi \tag{4.16}$$

and

$$\forall\varepsilon>0, \exists\delta>0 \text{ such that } f(x)\leqslant\delta\Rightarrow f(x)\geqslant(1-\varepsilon)\,\varphi(x). \tag{4.17}$$

Then formulas (4.9) *hold.*

The hypotheses (4.16) and (4.17) are verified, for instance, when f is C^2 and $\nabla^2 f(0)$ is a global minimum of the Hessian,

$$\forall x,y\in\mathbb{R}^n, \qquad \varphi(x):=\tfrac{1}{2}\nabla^2 f(0)\,x\cdot x\leqslant\tfrac{1}{2}\nabla^2 f(y)\,x\cdot x. \tag{4.18}$$

Then $f\geqslant\varphi$ and formulas (4.9) follow. Somewhat more generally, we may assume

$$\forall\varepsilon>0, \exists\delta>0, \forall x,y\in\mathbb{R}^n \qquad f(y)\leqslant\delta\Rightarrow(1-\varepsilon)\,\nabla^2 f(0)\,x\cdot x\leqslant\nabla^2 f(y)\,x\cdot x. \tag{4.19}$$

EXAMPLE 4.7. Consider the function on $\mathbb{R}^2$

$$f(x_1,x_2):=\sqrt{(x_1-1)^2+x_2^2}-(x_1-1). \tag{4.20}$$

It is a continuous, nonnegative convex function, whose graph is a convex cone in $\mathbb{R}^2$, with vertex $(-1,0,0)$ and intersecting the plane $x_3=0$ along the half-line $x_2=x_3=0$, $x_1\geqslant -1$. Of course f is C^∞ except at $(-1,0)$. The Hessian matrix of f at the origin is

$$\nabla^2 f(0,0)=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \tag{4.21}$$

and the second-order Taylor polynomial of f around $(0,0)$ is $(x_1, x_2) \mapsto x_2^2/2$. However,

$$\mathscr{E}_{(z_1, z_2), a} \leqslant f \Leftrightarrow \mathscr{E}_{(z_1, z_2), a} \equiv 0. \tag{4.22}$$

In fact, suppose that $(z_1, z_2) \neq (0, 0)$, $a > 0$ and

$$(z_1 x_1 + z_2 x_2)^2 \leqslant f(x_1, x_2) \qquad \forall (x_1, x_2) \text{ such that } 0 \leqslant z_1 x_1 + z_2 x_2 \leqslant a. \tag{4.23}$$

Evaluating the two sides of the inequality at $x_2 = 0$ we get $(z_1 x_1)^2 \leqslant 0$ for $0 \leqslant z_1 x_1 \leqslant a$, whence $z_1 = 0$. Next, evaluating at $(x_1, a/z_2)$, $x_1 \geqslant -1$, we have

$$0 < a^2 \leqslant \sqrt{(x_1 - 1)^2 + a^2} - (x_1 - 1) = \frac{a^2}{\sqrt{(x_1 - 1)^2 + a^2} + (x_1 - 1)}, \tag{4.24}$$

which is impossible, because the last term to the right vanishes as $x_1 \to +\infty$. In pictorial terms, the cone graph of f flattens out along the axis $x_2 = x_3 = 0$, as $x_1 \to +\infty$, squeezing to 0 whatever $\mathscr{E}_{z,a}$ whose graph happened to lie between the cone and the plane $x_3 = 0$. (This example is a reworking of one given by Seeger in [7].)

We can then say that, without positive definiteness of the Hessian, the C^2 (or even the C^∞) regularity does by no means ensure that the equality

$$\mathsf{D}^2_- f(0, 0; x) \equiv \tfrac{1}{2} \nabla^2 f(0) x \cdot x \tag{4.25}$$

holds (the "$\leqslant$" is always true, as well as the "$\equiv$" for D^2_+ instead of D^2_-, as it is easily seen). The trouble is that the value of the Hessian $\nabla^2 f(0)$ is a *local* property of f within the normed space structure, while $\partial^2_- f(0, 0)$ is a *global* linear space concept that, in a normed space, can be influenced by the behaviour of $f(x)$ for $|x|$ arbitrarily large.

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